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ON THE PROBLEM OF FLUID- AND GAS-FILLED ELASTO-PLASTIC SOLIDS†

R. DE BOER and W. EHLERS
Universität Essen, FB Bauwesen, Postfach 103 764, D-4300 Essen 1, West Germany

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Abstract—In the frame of continuum mechanics elasto-plastic porous solids with their intercommunicating void spaces filled with a viscous fluid and gas can be described as multicomponent models. For such models constitutive equations are developed, making full use of the thermodynamical restrictions. These constitutive equations are suitable to describe continua with ideal-plastic properties as well as brittle continua.

1. INTRODUCTION

The consideration of fluid- and gas-filled porous solids is of great importance in applied engineering as well as in the frame of soil mechanics and in describing other loaded porous structures, such as concrete.

In literature the theory of porous solids with void pores has been subject to an intensive research for a long time, e.g. publications by Berg[1], Caroll and Holt[2, 3], Green[4] and Shima and Oyane[5], whereas the theoretical basis of fluid- and gas-filled porous solids needs further clarification. A first approach to these problems has been done, e.g. by one of the others in [6].

The present paper concerns the theory of elasto-plastic fluid- and gas-filled porous continua, thus widening the plasticity theory for fluid-saturated porous solids given by de Boer and Kowalski[7].

In the frame of continuum mechanics elasto-plastic fluid- and gas-filled porous solids with their intercommunicating void spaces filled with a viscous fluid and gas can be described as multicomponent models by the introduction of volume fractions. The constituents of such models are a solid skeleton and free as well as trapped parts of the media fluid and gas.

The considered problems have a rather complex character, because the thermodynamical behaviour of the different media as well as the transport processes of fluid, gas and heat have to be analysed in general.

In the frame of the finite theory the problem generally leads to geometrically and physically nonlinear relations. In order to obtain a possible simple and practical theory several simplifications will be introduced. To this purpose a statistical distribution of the components solid skeleton, fluid and gas is assumed. Moreover, the different components are assumed to share a common temperature with a vanishing temperature gradient. Furthermore, the considered material is required to be isotropic. In addition, the compressibility of the components solid skeleton and fluid is assumed to be much smaller than the compressibility of the whole body. The stress deviators in the media fluid and gas are assumed to be negligible in comparison with the stress deviator in the skeleton. Finally, the development of constitutive equations in the frame of the finite theory is restricted to the case of small strains and finite rotations.

Considering the introduced limitation of the problem the balance equations and the fundamental equations of thermodynamics lead to a set of constitutive equations for elasto-plastic fluid- and gas-filled porous solids. These constitutive equations are suitable to describe continua with ideal-plastic properties as well as brittle continua. Concerning the vector and tensor calculus used in this paper, see [8].

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2. GENERAL CONSIDERATIONS

In order to state the problem consider a volume element of a porous solid (Fig. 1) which consists of an elasto-plastic solid skeleton with its intercommunicating and isolated void spaces filled with a viscous fluid and gas.

The pores of such volume elements are assumed to contain free as well as trapped parts of the media fluid and gas. The trapped parts, which can adhere as a molecular film on the side of the pores or may be trapped in isolated pore spaces share a common velocity with the solid skeleton, whereas the free parts can stream or diffuse through the open pore space. It is shown in literature[9] that classical diffusion and flow problems have a common mechanical basis, and that in several physical situations diffusion and flow processes do not affect the macroscopic response of the solid matrix. From a thermodynamical point of view these facts will also be a partial result of this paper. In the following we will not study the microstructure of the considered medium but treat the mentioned problems in the frame of continuum mechanics by introducing a multicomponent model. To this purpose it is convenient to use the indication

$(\ldots)^s$:	solid skeleton,
$(\ldots)^{Ff}$ and $(\ldots)^{Ft}$:	free and trapped fluid,
$(\ldots)^{Gf}$ and $(\ldots)^{Gt}$:	free and trapped gas,

in order to distinguish the mechanical quantities of the different components.

As usual in classical mechanics one chooses a material description for the solid skeleton and the trapped parts of fluid and gas

$$\begin{aligned} \mathbf{x} &= \hat{\chi}(\mathbf{X}, t), \\ \mathbf{v} &= \dot{\hat{\chi}}(\mathbf{X}, t), \end{aligned} \tag{2.1}$$

whereas the components free fluid and free gas are described by their actual velocities

$$\begin{aligned} \mathbf{w}^F &= \hat{\mathbf{w}}^F(\mathbf{x}, t), \\ \mathbf{w}^G &= \hat{\mathbf{w}}^G(\mathbf{x}, t) \end{aligned} \tag{2.2}$$

with the position vectors $\mathbf{x}$ of the actual and $\mathbf{X}$ of the reference configuration and the time t.

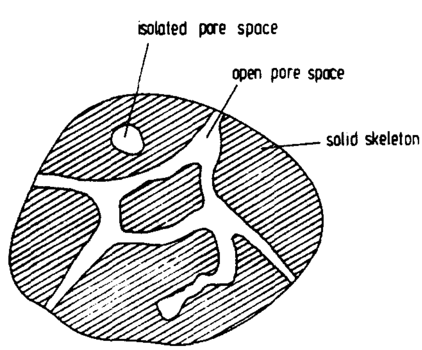

Fig. 1. Basic structure of a porous volume element; pores filled with a viscous fluid and gas.

Owing to the sharing of a common velocity by the skeleton and the trapped parts of fluid and gas, these components can be regarded as a comprised continuum so that the considered multicomponent model consists of three partial continua: the solid skeleton including the trapped parts of fluid and gas, the free fluid and the free gas. Due to the different velocity states, different material time derivatives with different convective parts have to be considered:

$$
\begin{aligned}
(\ldots)^{\cdot} &= \frac{\partial(\ldots)}{\partial t} + \operatorname{grad}(\ldots)\cdot \mathbf{v}, \\
(\ldots)' &= \frac{\partial(\ldots)}{\partial t} + \operatorname{grad}(\ldots)\cdot \mathbf{w}^F, \\
(\ldots)^{\nabla} &= \frac{\partial(\ldots)}{\partial t} + \operatorname{grad}(\ldots)\cdot \mathbf{w}^G.
\end{aligned}
\tag{2.3}
$$

Assuming a statistical distribution of the media solid skeleton, fluid and gas inside the considered multi-component model, the demand for the existence of volume fractions

$$
\begin{aligned}
&n^S; \quad n^{Ff}; \quad n^{Ft}; \quad n^{Gf}; \quad n^{Gt}; \\
&n^F = n^{Ff} + n^{Ft}; \qquad n^G = n^{Gf} + n^{Gt}
\end{aligned}
\tag{2.4}
$$

leads to the following representation of the volume V as the sum of the partial volumes of the particular components:

$$
\begin{aligned}
V &= \int_p \mathrm{d}v \\
&= \int_p (\mathrm{d}v^S + \mathrm{d}v^{Ff} + \mathrm{d}v^{Ft} + \mathrm{d}v^{Gf} + \mathrm{d}v^{Gt}) \\
&= \int_p (n^S + n^{Ff} + n^{Ft} + n^{Gf} + n^{Gt})\, \mathrm{d}v.
\end{aligned}
\tag{2.5}
$$

The volume fractions which are assumed to be sufficiently smooth in space-time are connected with the classical quantities volume porosity n^v (equal to the average surface porosity) and effective surface porosity n^{su}, see, e.g. [10]:

$$
\begin{aligned}
n^v &= n^F + n^G, \\
n^{su} &= n^{Ff} + n^{Gf}.
\end{aligned}
\tag{2.6}
$$

With the true mass densities ρ_R^S, ρ_R^F, ρ_R^G of the skeleton, the fluid and the gas and the relations (2.4) and (2.5), the bulk densities with respect to the total volume element $\mathrm{d}v$ can be introduced:

$$
\begin{aligned}
&\rho^S = n^S \rho_R^S, \\
&\rho^{Ff} = n^{Ff}\rho_R^F; \qquad \rho^{Ft} = n^{Ft}\rho_R^F; \qquad \rho^F = n^F\rho_R^F, \\
&\rho^{Gf} = n^{Gf}\rho_R^G; \qquad \rho^{Gt} = n^{Gt}\rho_R^G; \qquad \rho^G = n^G\rho_R^G.
\end{aligned}
\tag{2.7}
$$

Considering the preceding relations stated additionally to those of the classical continuum mechanics, the thermomechanical behaviour of fluid- and gas-filled porous solids can be specified in the following.

3. BALANCE EQUATIONS

3.1. *Conservation of mass*

Within the formulation of mass conservation laws it is assumed that the particular parts of the total mass—solid skeleton, fluid and gas—do not change during the motion the body is subject to. For the solid skeleton this assumption is obvious and leads to the well-known local form

$$0 = \dot{\rho}^S + \rho^S \operatorname{div} \mathbf{v}. \tag{3.1.1}$$

For the media fluid and gas, however, this requirement of mass conservation excludes any mass sources inside the body.

These simplifications lead to the conservation laws

$$\begin{aligned} 0 &= \dot{\rho}^F + \rho^F \operatorname{div} \mathbf{v} + \operatorname{div} [\rho^{Ff}(\mathbf{w}^F - \mathbf{v})], \\ 0 &= \dot{\rho}^G + \rho^G \operatorname{div} \mathbf{v} + \operatorname{div} [\rho^{Gf}(\mathbf{w}^G - \mathbf{v})]. \end{aligned} \tag{3.1.2}$$

Assuming the solid skeleton and the fluid to be homogeneous and to be incompressible in comparison with the whole body, the relations

$$\begin{aligned} &\frac{\partial \rho_R^S}{\partial t} \equiv 0, \qquad \operatorname{grad} \rho_R^S \equiv \mathbf{o}, \\ &\frac{\partial \rho_R^F}{\partial t} \equiv 0, \qquad \operatorname{grad} \rho_R^F \equiv \mathbf{o} \end{aligned} \tag{3.1.3}$$

are valid.

From (2.4) and (2.5) follows

$$1 = n^S + n^F + n^G \tag{3.1.4}$$

so that (2.7) and (3.1.1)–(3.1.4) lead to the material time derivatives of the introduced volume fractions

$$\begin{aligned} \dot{n}^S &= -n^S \operatorname{div} \mathbf{v}, \\ \dot{n}^F &= -n^F \operatorname{div} \mathbf{v} - \operatorname{div} [n^{Ff}(\mathbf{w}^F - \mathbf{v})], \\ \dot{n}^G &= (1 - n^G) \operatorname{div} \mathbf{v} + \operatorname{div} [n^{Ff}(\mathbf{w}^F - \mathbf{v})], \end{aligned} \tag{3.1.5}$$

which are needed to describe incremental loading processes.

3.2. *Equations of motion*

For the defined three partial continua of the considered multicomponent model different balance equations of momentum and of moment of momentum have to be stated.

Following this, Cauchy's first law of motion for the skeleton including the trapped parts of fluid and gas, for the free fluid and for the free gas, yields with respect to the total volume element dv

$$\begin{aligned} \operatorname{div} \mathbf{T}^S - \operatorname{grad} p^{Ft} - \operatorname{grad} p^{Gt} + (\rho^S + \rho^{Ft} + \rho^{Gt})(\mathbf{b} - \dot{\mathbf{v}}) + \mathbf{f}^{SF} + \mathbf{f}^{SG} & \\ = (\dot{\rho}^{Ft} + \rho^{Ft} \operatorname{div} \mathbf{v} + \dot{\rho}^{Gt} + \rho^{Gt} \operatorname{div} \mathbf{v})\mathbf{v}&, \\ -\operatorname{grad} p^{Ff} + \rho^{Ff}(\mathbf{b} - \mathbf{w}'^F) - \mathbf{f}^{SF} = (\rho'^{Ff} + \rho^{Ff} \operatorname{div} \mathbf{w}^F)\mathbf{w}^F&, \\ -\operatorname{grad} p^{Gf} + \rho^{Gf}(\mathbf{b} - \mathbf{w}^{\nabla G}) - \mathbf{f}^{SG} = (\rho^{\nabla Gf} + \rho^{Gf} \operatorname{div} \mathbf{w}^G)\mathbf{w}^G&. \end{aligned} \tag{3.2.1}$$

In these equations, in which the deviatoric parts of the stress states in the media fluid and gas as well as the viscous properties inside the fluid are neglected—an admissible simplification for physical situations with sufficiently slow flow processes—$\mathbf{b}$ denotes the body forces per unit mass and $\mathbf{T}^S = n^S\mathbf{T}^S_R$ (with the true Cauchy stress $\mathbf{T}^S_R$) Cauchy's stress tensor in the solid skeleton defined as the sum of the stress tensor $\bar{\mathbf{T}}^S$ and the partial hydrostatic pressure imposed on the skeleton:

$$\mathbf{T}^S = \bar{\mathbf{T}}^S - n^S p\mathbf{I}. \tag{3.2.2}$$

In soil mechanics $\bar{\mathbf{T}}^S$ is usually defined as the stress tensor caused by contact forces between the grains of granular materials. The symmetry $\mathbf{T}^S = (\mathbf{T}^S)^T$ follows from Cauchy's second law of motion, assuming that there exists no moment of momentum supply in contrast to the usual definitions in the theory of mixtures[11]. The quantities p^{Ff}, p^{Ft}, p^{Gf} and p^{Gt} signify the relative hydrostatic pressures of the free and trapped parts of the media fluid and gas:

$$p^{Ff} = n^{Ff}p, \qquad \text{etc.} \tag{3.2.3}$$

The momentum supplies $\mathbf{f}^{SF}$ and $\mathbf{f}^{SG}$ represent the interacting forces caused by friction and diffusion effects between the skeleton, including the trapped parts of fluid and gas and the free fluid or the free gas, respectively. An interacting force between the free fluid and the free gas is assumed to be small and therefore negligible. Capillary stresses which might be treated as surface forces are also neglected in this paper, but see, e.g. [6].

4. THERMODYNAMICAL RESTRICTIONS

The partial continua of the considered multicomponent model are assumed to be thermodynamical units with a mutual exchange of heat and work. Therefore, the energy balance equations for the solid skeleton with the trapped parts of fluid and gas, for the free fluid and the free gas, are

$$\begin{aligned} \dot{E}+\dot{K} &= \dot{W}+\dot{Q}+\int_P e\,\mathrm{d}v, \\ E'^{Ff}+K'^{Ff} &= W'^{Ff}+Q'^{Ff}+\int_P e^{Ff}\,\mathrm{d}v, \\ E^{\nabla Gf}+K^{\nabla Gf} &= W^{\nabla Gf}+Q^{\nabla Gf}+\int_P e^{Gf}\,\mathrm{d}v, \end{aligned} \tag{4.1}$$

where E denotes the internal energy, K the kinetic energy, W the mechanical work, Q the heat and e the local energy supply terms of the partial continua satisfying the condition $e+e^{Ff}+e^{Gf} = 0$. The energy balance equation of the total continuum yields from the sum of eqns (4.1) and will be restricted to those physical situations for which the following assumptions are valid:

(1) the partial continua have an identical temperature state,

(2) there exists a homogeneous distribution of the absolute temperature θ: grad $\theta = \mathbf{o}$,

(3) there exists no exchange between the free and trapped parts of the media fluid and gas so that the mass continuity equations (3.1.2) become

$$\begin{aligned} 0 &= \rho'^{Ff}+\rho^{Ff}\operatorname{div}\mathbf{w}^F; \qquad & 0 &= \dot{\rho}^{Ft}+\rho^{Ft}\operatorname{div}\mathbf{v}, \\ 0 &= \rho^{\nabla Gf}+\rho^{Gf}\operatorname{div}\mathbf{w}^G; \qquad & 0 &= \dot{\rho}^{Gt}+\rho^{Gt}\operatorname{div}\mathbf{v}. \end{aligned} \tag{4.2}$$

Following this, the local form of the energy balance equation of the total continuum is

$$\begin{aligned}\rho^S\dot{\varepsilon}^S+\rho^F\dot{\varepsilon}^F+\rho^G\dot{\varepsilon}^G+\rho^{Ff}\operatorname{grad}\varepsilon^{Ff}\cdot(\mathbf{w}^F-\mathbf{v})+\rho^{Gf}\operatorname{grad}\varepsilon^{Gf}\cdot(\mathbf{w}^G-\mathbf{v})\\ =\mathbf{\mathring{T}}\cdot\mathbf{D}-p^{Ff}\operatorname{div}\mathbf{w}^F-p^{Gf}\operatorname{div}\mathbf{w}^G+\mathbf{f}^{SF}\cdot(\mathbf{w}^F-\mathbf{v})\\ +\mathbf{f}^{SG}\cdot(\mathbf{w}^G\cdot\mathbf{v})+\rho^Sr^S+\rho^Fr^F+\rho^Gr^G-\operatorname{div}(\mathbf{q}^S+\mathbf{q}^F+\mathbf{q}^G)\end{aligned} \quad (4.3)$$

with

$$\mathbf{\mathring{T}}=\mathbf{T}^S-(p^{Ft}+p^{Gt})\mathbf{I},\qquad \mathbf{D}=\tfrac{1}{2}(\operatorname{grad}\mathbf{v}+\operatorname{grad}^T\mathbf{v}), \quad (4.4)$$

where ε is the specific internal energy, r the supply of heat and $\mathbf{q}$ the heat flux vector of the particular components, each quantity referred to the total volume element $\mathrm{d}v$. Besides the classical terms rate of internal energy, rate of mechanical work and thermal quantities the influence of the work achieved by the interacting forces enters the energy balance equation.

The entropy inequality for the total continuum is

$$\dot{H}+H'^{Ff}+H^{\nabla Gf}\geqslant\int_p\frac{1}{\theta}(\rho^Sr^S+\rho^Fr^F+\rho^Gr^G)\,\mathrm{d}v-\int_{\partial p}\frac{1}{\theta}(\mathbf{q}^S+\mathbf{q}^F+\mathbf{q}^G)\cdot\mathrm{d}\mathbf{a} \quad (4.5)$$

with the entropies

$$\begin{aligned}H&=\int_p(\rho^S\eta^S+\rho^{Ft}\eta^F+\rho^{Gt}\eta^G)\,\mathrm{d}v,\\ H^{Ff}&=\int_p\rho^{Ff}\eta^F\,\mathrm{d}v,\\ H^{Gf}&=\int_p\rho^{Gf}\eta^G\,\mathrm{d}v.\end{aligned} \quad (4.6)$$

In (4.6), η denotes the specific entropy of the particular components.

Using the specific free energy function ψ for the particular components the inequality (4.5) yields

$$\begin{aligned}&-\rho^S(\dot{\psi}^S+\dot{\theta}\eta^S)-\rho^F(\dot{\psi}^F+\dot{\theta}\eta^F)-\rho^G(\dot{\psi}^G+\dot{\theta}\eta^G)\\ &-\rho^{Ff}\operatorname{grad}\psi^{Ff}\cdot(\mathbf{w}^F-\mathbf{v})-\rho^{Gf}\operatorname{grad}\psi^{Gf}\cdot(\mathbf{w}^G-\mathbf{v})\\ &+\mathbf{\mathring{T}}\cdot\mathbf{D}-p^{Ff}\operatorname{div}\mathbf{w}^F-p^{Gf}\operatorname{div}\mathbf{w}^G\\ &+\mathbf{f}^{SF}\cdot(\mathbf{w}^F-\mathbf{v})+\mathbf{f}^{SG}\cdot(\mathbf{w}^G-\mathbf{v})\geqslant 0,\end{aligned} \quad (4.7)$$

where the quantities div $\mathbf{w}^F$ and div $\mathbf{w}^G$ can be replaced by the mass continuity equations (3.1.2).

Assuming the decomposition of the stretching $\mathbf{D}$ and the corresponding strain rate $\dot{\mathbf{E}}$ into an elastic and an inelastic part

$$\begin{aligned}\mathbf{D}&=\mathbf{D}_e+\mathbf{D}_i,\\ \dot{\mathbf{E}}&=\dot{\mathbf{E}}_e+\dot{\mathbf{E}}_i\qquad\text{with}\quad \dot{\mathbf{E}}_e=\mathbf{F}^T\mathbf{D}_e\mathbf{F},\\ &\qquad\qquad\qquad\qquad\quad\ \dot{\mathbf{E}}_i=\mathbf{F}^T\mathbf{D}_i\mathbf{F},\end{aligned} \quad (4.8)$$

(4.7) yields

$$
\begin{aligned}
&-\rho^S(\dot{\psi}^S+\dot{\theta}\eta^S)-\rho^F(\dot{\psi}^F+\dot{\theta}\eta^F)-\rho^G(\dot{\psi}^G+\dot{\theta}\eta^G)\\
&+\left(\frac{p^{Ff}}{\rho^{Ff}}\operatorname{grad}\rho^{Ff}-\rho^{Ff}\operatorname{grad}\psi^{Ff}\right)\cdot(\mathbf{w}^F-\mathbf{v})\\
&+\left(\frac{p^{Gf}}{\rho^{Gf}}\operatorname{grad}\rho^{Gf}-\rho^{Gf}\operatorname{grad}\psi^{Gf}\right)\cdot(\mathbf{w}^G-\mathbf{v})\\
&+\mathbf{T}^S\cdot\mathbf{D}_e+\mathbf{T}^S\cdot\mathbf{D}_i+p^F\frac{\dot{\rho}^F}{\rho^F}+p^G\frac{\dot{\rho}^G}{\rho^G}\\
&+\mathbf{f}^{SF}\cdot(\mathbf{w}^F-\mathbf{v})+\mathbf{f}^{SG}\cdot(\mathbf{w}^G-\mathbf{v})\geqslant 0.
\end{aligned}
\tag{4.9}
$$

This form of the entropy inequality can be used as a thermodynamical restriction for the development of constitutive equations of the considered multicomponent medium.

5. CONSTITUTIVE EQUATIONS

5.1. *General relations*

Within the formulation of constitutive assumptions in the frame of the finite theory the solid skeleton is assumed to be isotropic with a dissipating part of the free energy as a result of plastic deformations and of flow and diffusion processes. Then, the increase of the temperature can be a measure for the energy dissipation. Considering Noll's postulates determinism, local action and material frame-indifference it can be shown (see, e.g. [12]) that the specific free energy of the skeleton is a function of the invariants of the elastic part of Green's deformation tensor $\mathbf{B}$ and the temperature θ as far as other internal variables are assumed to be negligible:

$$
\begin{aligned}
&\psi^S=\hat{\psi}^S(\mathbf{B}_e,\theta)=\hat{\psi}^S(I_{\mathbf{B}_e},II_{\mathbf{B}_e},III_{\mathbf{B}_e},\theta),\\
&\dot{\psi}^S=\frac{\partial\hat{\psi}^S(\mathbf{B}_e,\theta)}{\partial\theta}\dot{\theta}+\left[\frac{\partial\hat{\psi}^S(\mathbf{B}_e,\theta)}{\partial\mathbf{B}_e}+\left(\frac{\partial\hat{\psi}^S(\mathbf{B}_e,\theta)}{\partial\mathbf{B}_e}\right)^T\right]\mathbf{B}_e\cdot\mathbf{D}_e.
\end{aligned}
\tag{5.1.1}
$$

In this paper the specific free energies of the media fluid and gas are assumed to be a function of the temperature state and the bulk densities only, thereby neglecting the viscous properties inside the fluid as well as other internal variables which might be stated to describe constitutive interaction between the particular components. The consideration of such variables is possible but is not discussed in this paper:

$$
\begin{aligned}
&\psi^F=\hat{\psi}^F(\theta,\rho^F); &&\dot{\psi}^F=\frac{\partial\hat{\psi}^F(\theta,\rho^F)}{\partial\theta}\dot{\theta}+\frac{\partial\hat{\psi}^F(\theta,\rho^F)}{\partial\rho^F}\dot{\rho}^F,\\
&\psi^G=\hat{\psi}^G(\theta,\rho^G); &&\dot{\psi}^G=\frac{\partial\hat{\psi}^G(\theta,\rho^G)}{\partial\theta}\dot{\theta}+\frac{\partial\hat{\psi}^G(\theta,\rho^G)}{\partial\rho^G}\dot{\rho}^G
\end{aligned}
\tag{5.1.2}
$$

and

$$
\begin{aligned}
&\psi^{Ff}=\hat{\psi}^{Ff}(\theta,\rho^{Ff}); &&\operatorname{grad}\psi^{Ff}=\frac{\partial\hat{\psi}^{Ff}(\theta,\rho^{Ff})}{\partial\rho^{Ff}}\operatorname{grad}\rho^{Ff},\\
&\psi^{Gf}=\hat{\psi}^{Gf}(\theta,\rho^{Gf}); &&\operatorname{grad}\psi^{Gf}=\frac{\partial\hat{\psi}^{Gf}(\theta,\rho^{Gf})}{\partial\rho^{Gf}}\operatorname{grad}\rho^{Gf}.
\end{aligned}
\tag{5.1.3}
$$

With the help of (5.1.1)–(5.1.3) the inequality (4.9) becomes

$$\left\{-\rho^S\left[\frac{\partial\psi^S}{\partial\mathbf{B}_e}+\left(\frac{\partial\psi^S}{\partial\mathbf{B}_e}\right)^T\right]\mathbf{B}_e+\mathbf{T}^S\right\}\cdot\mathbf{D}_e-\rho^S\left(\frac{\partial\psi^S}{\partial\theta}+\eta^S\right)\dot{\theta}$$
$$+\left(\frac{p^F}{\rho^F}-\rho^F\frac{\partial\psi^F}{\partial\rho^F}\right)\dot{\rho}^F-\rho^F\left(\frac{\partial\psi^F}{\partial\theta}+\eta^F\right)\dot{\theta}$$
$$+\left(\frac{p^G}{\rho^G}-\rho^G\frac{\partial\psi^G}{\partial\rho^G}\right)\dot{\rho}^G-\rho^G\left(\frac{\partial\psi^G}{\partial\theta}+\eta^G\right)\dot{\theta}$$
$$+\left(\frac{p^{Ff}}{\rho^{Ff}}-\rho^{Ff}\frac{\partial\psi^{Ff}}{\partial\rho^{Ff}}\right)\operatorname{grad}\rho^{Ff}\cdot(\mathbf{w}^F-\mathbf{v})$$
$$+\left(\frac{p^{Gf}}{\rho^{Gf}}-\rho^{Gf}\frac{\partial\psi^{Gf}}{\partial\rho^{Gf}}\right)\operatorname{grad}\rho^{Gf}\cdot(\mathbf{w}^G-\mathbf{v})$$
$$+\mathbf{T}^S\cdot\mathbf{D}_i+\mathbf{f}^{SF}\cdot(\mathbf{w}^F-\mathbf{v})+\mathbf{f}^{SG}\cdot(\mathbf{w}^G-\mathbf{v})\geqslant 0. \tag{5.1.4}$$

Equation (5.1.4) must hold in each point for any arbitrary process. Then, from (5.1.4) the following constitutive relations are valid:

$$\mathbf{T}^S=\rho^S\left[\frac{\partial\psi^S}{\partial\mathbf{B}_e}+\left(\frac{\partial\psi^S}{\partial\mathbf{B}_e}\right)^T\right]\mathbf{B}_e,$$
$$\rho^S\eta^S+\rho^F\eta^F+\rho^G\eta^G=-\left(\rho^S\frac{\partial\psi^S}{\partial\theta}+\rho^F\frac{\partial\psi^F}{\partial\theta}+\rho^G\frac{\partial\psi^G}{\partial\theta}\right),$$
$$p^F=(\rho^F)^2\frac{\partial\psi^F}{\partial\rho^F};\qquad p^G=(\rho^G)^2\frac{\partial\psi^G}{\partial\rho^G},$$
$$p^{Ff}=(\rho^{Ff})^2\frac{\partial\psi^{Ff}}{\partial\rho^{Ff}};\qquad p^{Gf}=(\rho^{Gf})^2\frac{\partial\psi^{Gf}}{\partial\rho^{Gf}}. \tag{5.1.5}$$

Using (5.1.5) the rate of dissipative energy as a reduced form of the entropy inequality (5.1.4) yields

$$\mathbf{T}^S\cdot\mathbf{D}_i+\mathbf{f}^{SF}\cdot(\mathbf{w}^F-\mathbf{v})+\mathbf{f}^{SG}\cdot(\mathbf{w}^G-\mathbf{v})\geqslant 0. \tag{5.1.6}$$

From (5.1.6) the following restrictions for the rate of inelastic energy and the interacting forces hold:

$$\mathbf{T}^S\cdot\mathbf{D}_i\geqslant 0\qquad\text{resp.}\quad\mathbf{S}^S\cdot\dot{\mathbf{E}}_i\geqslant 0,$$
$$\mathbf{f}^{SF}\cdot(\mathbf{w}^F-\mathbf{v})\geqslant 0, \tag{5.1.7}$$
$$\mathbf{f}^{SG}\cdot(\mathbf{w}^G-\mathbf{v})\geqslant 0.$$

$\mathbf{S}^S$ denotes the second Piola–Kirchhoff stress tensor of the skeleton corresponding to $\mathbf{T}^S$. By satisfying the last two restrictions sufficiently linear constitutive relations for the interacting forces can be found:

$$\mathbf{f}^{SF}=\mathbf{A}(\mathbf{w}^F-\mathbf{v}),\qquad\mathbf{A}\text{ positive definite},$$
$$\mathbf{f}^{SG}=\mathbf{B}(\mathbf{w}^G-\mathbf{v}),\qquad\mathbf{B}\text{ positive definite}. \tag{5.1.8}$$

The quantities **A** and **B** are material tensors. Omitting the body forces the insertion of (5.1.8) and (4.2) into $(3.2.1)_2$ resp. $(3.2.1)_3$ yields in case of isotropic mechanical behaviour the general forms of Darcy's filter law and Fick's first diffusion law,

$$\mathbf{w}^F - \mathbf{v} = -\frac{1}{a}\operatorname{grad} p^{Ff}, \qquad a \geqslant 0,$$
$$\mathbf{w}^G - \mathbf{v} = -\frac{1}{b}\operatorname{grad} p^{Gf}, \qquad b \geqslant 0, \tag{5.1.9}$$

with the material parameters a and b.

From these representations it is clear that classical resp. linear flow and diffusion problems have a common mechanical basis and considering the preceding constitutive assumptions flow and diffusion processes do not affect the response of the skeleton; see, e.g. [9].

For a homogeneous distributed (grad $n^{Ff} = \mathbf{o}$) free fluid Darcy's law becomes

$$\mathbf{w}^F - \mathbf{v} = k\mathbf{i}, \qquad k = \frac{\gamma^{Ff}}{a}, \qquad \mathbf{i} = -\operatorname{grad} h, \tag{5.1.10}$$

where k denotes the coefficient of permeability, $\mathbf{i}$ the hydraulic gradient, γ^{Ff} the relative specific weight and h the pressure head of the free fluid.

Fick's first diffusion law for an ideal free gas can be written as

$$\mathbf{j} = -\hat{D}(\rho^{Gf}) \operatorname{grad} \rho^{Gf}, \qquad \mathbf{j} = \rho^{Gf}(\mathbf{w}^G - \mathbf{v}), \qquad \hat{D}(\rho^{Gf}) = \frac{\bar{R}\theta\rho^{Gf}}{b}, \tag{5.1.11}$$

with the diffusive flux vector $\mathbf{j}$ and the diffusivity D, in which $\bar{R}$ is the gas constant. Fick's second diffusion law

$$\frac{\partial \rho^{Gf}}{\partial t} = D\Delta\rho^{Gf} \qquad (\Delta\text{: Laplace operator}) \tag{5.1.12}$$

results from (5.1.11) and the mass continuity equation for the component free gas (see (4.2)) assuming D to be a constant and the velocity $\mathbf{v}$ to be identically zero.

5.2. *Elastic material properties*

The general constitutive equation $(5.1.5)_1$ for an isotropic solid skeleton can be represented by the theorem

$$\mathbf{T}^S = 2\rho^S(\varphi_0\mathbf{I} + \varphi_1\mathbf{B} + \varphi_2\mathbf{BB}), \tag{5.2.1}$$

see [13], where the coefficients φ_0, φ_1 and φ_2 are isotropic functions of $\mathbf{B}$. In this representation, valid in frame of the finite theory, the included material properties cannot be specified sufficiently by tensile and shear tests to describe three dimensional material behaviour. Therefore, the following is restricted to the theory of small strains and finite rotations. Within this theory the free energy ψ^S of the skeleton can be developed near the natural state, which is taken as the undistorted reference configuration, into Taylor's series.

The result of this procedure (see e.g. [12]) are the constitutive equations

$$\mathbf{T}^S = \frac{1}{\det \mathbf{F}}\mathbf{F}[\overset{4}{\mathbf{B}}_e\mathbf{E} + \mathbf{G}_e(\theta - \theta_0)]\mathbf{F}^T,$$
$$\mathbf{S}^S = \overset{4}{\mathbf{B}}_e\mathbf{E} + \mathbf{G}_e(\theta - \theta_0), \tag{5.2.2}$$

where θ_0 is the temperature of the reference state.

The rate equation

$$\dot{\mathbf{S}}^S = \overset{4}{\mathbf{B}}_e \dot{\mathbf{E}} + \mathbf{G}_e \dot{\theta} \tag{5.2.3}$$

is needed to describe elasto-plastic material properties in terms of the reference configuration. The fourth order material tensor $\overset{4}{\mathbf{B}}_e$ and the material tensor $\mathbf{G}_e$ are defined by

$$\begin{aligned} \overset{4}{\mathbf{B}}_e &= n_0^S \overset{4}{\bar{\mathbf{B}}}_e ; \quad & \overset{4}{\bar{\mathbf{B}}}_e &= 2\mu \overset{4}{\mathbf{I}}_e + \lambda(\mathbf{I} \otimes \mathbf{I}), \\ \mathbf{G}_e &= n_0^S \bar{\mathbf{G}}_e ; \quad & \bar{\mathbf{G}}_e &= \alpha_t \mathbf{I}. \end{aligned} \tag{5.2.4}$$

In (5.2.4), n_0^S stands for the volume fraction of the skeleton in the natural state, $\overset{4}{\bar{\mathbf{B}}}_e$ for Hooke's law with the Lamé constants μ and λ and α_t for the coefficient of thermal expansion. The quantity $\overset{4}{\mathbf{I}}$ represents the fourth order identity.

5.3. *Plastic material properties*

In order to describe ideal-plastic or brittle material properties of fluid- and gas-filled porous solids the scalar-valued tensor function

$$\phi = \sqrt{J_2 + \tfrac{1}{2}\alpha^2(\mathbf{S}^S \cdot \mathbf{I})^2} + \beta(\mathbf{S}^S \cdot \mathbf{I}) - \kappa = 0 \tag{5.3.1}$$

can be introduced representing a yield as well as a failure condition for the considered material. This equation, in which J_2 denotes the second invariant of the stress deviator of the skeleton, contains the positive material parameters α, β and κ, which serve to adapt the yield condition to the experimental results.

The proposed yield condition contains several special cases:

(1) If α and β are equal to zero, (5.3.1) turns into the well-known yield condition of von Mises.

(2) If only β is equal to zero, (5.3.1) changes into a condition which is suitable to describe the yield point of ductile materials with volume strains in the plastic region. Such a condition has been proposed by de Boer and Kowalski[7] to describe the yield point of fluid-saturated porous solids.

(3) If only α is equal to zero, (5.3.1) yields the form proposed by Drucker and Prager[14], which represents a generalization of Mohr's and Coulomb's condition in soil mechanics.

At this point it should be noticed that using (5.3.1) the solid matrix of the considered material must stay coherent without any separating fracture as far as the methods of the plasticity theory shall be used. This holds for ductile as well as for many granular and brittle media. For example, during the failure process sandy-like soils show certain strata in which the grains are rolling one upon the other. During the loading process of brittle materials like concrete, there first occur single cracks in several regions. These cracks widen until the failure of the structure takes place, whereby the occurring failure regions are similar to those of soils.

Furthermore, it is worth noting that (5.3.1) is of hyperbolic type, which can easily be shown by writing this condition as a function of $\sqrt{J_2}$ as far as the condition $\beta^2 - \frac{1}{2}\alpha^2 > 0$ is fulfilled:

$$\sqrt{J_2} = \sqrt{((\beta^2 - \tfrac{1}{2}\alpha^2)(\mathbf{S}^S \cdot \mathbf{I})^2 - 2\beta\kappa(\mathbf{S}^S \cdot \mathbf{I}) + \kappa^2)}. \tag{5.3.2}$$

In order to give an example for the validity of the proposed yield function consider Fig. 2, which shows the failure condition of concrete established by experiments of Mills and Zimmermann[15]. This curve is referred to the prism strength incompression β_p of concrete, β_p being a function of the concrete strength as well as the porosity and the shape of the respective specimens. Thus, Fig. 2 is independent of these quantities.

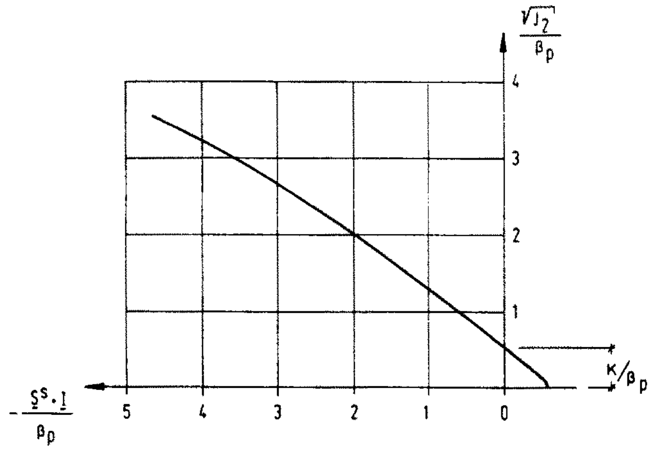

Fig. 2. Failure condition for concrete; material parameters $\alpha = 0.05$, $\beta = 0.68$, $\kappa/\beta_p = 0.47$.

Determining the material constants $\alpha = 0.05$, $\beta = 0.68$ and $\kappa/\beta_p = 0.47$, Fig. 2 can be approximated nearly identically with the proposed failure function.

According to the concept of plastic potential the inelastic strain rate $\dot{\mathbf{E}}_i$ corresponding to the yield condition (5.3.1) yields in case of ideal-plastic materials

$$\dot{\mathbf{E}}_i = \dot{\lambda}\frac{\partial\phi}{\partial\mathbf{S}^S},$$

$$\frac{\partial\phi}{\partial\mathbf{S}^S} = \frac{\mathbf{S}^{SD}+\alpha^2(\mathbf{S}^S\cdot\mathbf{I})\mathbf{I}}{2\sqrt{(J_2+\frac{1}{3}\alpha^2(\mathbf{S}^S\cdot\mathbf{I})^2)}}+\beta\mathbf{I}; \qquad \mathbf{S}^{SD} = \mathbf{S}^S-\tfrac{1}{3}(\mathbf{S}^S\cdot\mathbf{I})\mathbf{I} \tag{5.3.3}$$

with the loading criteria

$$\phi = 0 \text{ and } \frac{\partial\phi}{\partial\mathbf{S}^S}\cdot\dot{\mathbf{S}}^S\begin{cases}=0; & \text{neutral state}; \quad \dot{\mathbf{E}}_i \neq \mathbf{0},\\ <0: & \text{unloading}; \quad \dot{\mathbf{E}}_i = \mathbf{0}.\end{cases} \tag{5.3.4}$$

From (5.3.1), (5.3.3) and the restriction $(5.1.7)_1$ $\dot{\lambda}$ can be specified to be a positive scalar-valued function in the neutral state (loading for ideal-plastic solids) or to be zero in case of unloading:

$$\mathbf{S}^S\cdot\dot{\mathbf{E}}_i = \dot{\lambda}\kappa \geqslant 0 \rightarrow \dot{\lambda} \geqslant 0. \tag{5.3.5}$$

Expressing $\dot{\lambda}$ by the material parameters α and β and the inelastic strain rates one obtains:

$$\dot{\lambda} = \frac{2\beta}{3(2\beta^2-\alpha^2)}\dot{\mathbf{E}}_i\cdot\mathbf{I}-\sqrt{\left(\frac{2\alpha^2}{2\beta^2-\alpha^2}\left[\frac{(\dot{\mathbf{E}}_i\cdot\mathbf{I})^2}{9(2\beta^2-\alpha^2)}-\dot{\mathbf{E}}_i^D\cdot\dot{\mathbf{E}}_i^D\right]\right)};$$

$$\dot{\mathbf{E}}_i^D = \dot{\mathbf{E}}_i-\tfrac{1}{3}(\dot{\mathbf{E}}_i\cdot\mathbf{I})\mathbf{I}. \tag{5.3.6}$$

Equation (5.3.6) represents an important expression, especially with regard to the theory of limit design.

5.4. *Elasto-plastic material properties*

In order to describe elasto-plastic material properties of the solid skeleton we take the usual way (see, e.g. [12]), deriving the constitutive equation with respect to the decomposition of the strain rates (4.8).

The result of this procedure is

$$\dot{\mathbf{S}}^S = \overset{4}{\mathbf{B}}\dot{\mathbf{E}}+\mathbf{G}\dot{\theta} \tag{5.4.1}$$

with

$$\overset{4}{\mathbf{B}} = \overset{4}{\mathbf{B}}_e - \frac{\overset{4}{\mathbf{B}}_e \dfrac{\partial\phi}{\partial\mathbf{S}^S} \otimes \overset{4}{\mathbf{B}}_e \dfrac{\partial\phi}{\partial\mathbf{S}^S}}{\overset{4}{\mathbf{B}}_e \cdot \left(\dfrac{\partial\phi}{\partial\mathbf{S}^S} \otimes \dfrac{\partial\phi}{\partial\mathbf{S}^S}\right)},$$
$$\mathbf{G} = \mathbf{G}_e - \frac{\overset{4}{\mathbf{B}}_e \dfrac{\partial\phi}{\partial\mathbf{S}^S} \left(\mathbf{G}_e \cdot \dfrac{\partial\phi}{\partial\mathbf{S}^S}\right)}{\overset{4}{\mathbf{B}}_e \cdot \left(\dfrac{\partial\phi}{\partial\mathbf{S}^S} \otimes \dfrac{\partial\phi}{\partial\mathbf{S}^S}\right)}, \tag{5.4.2}$$

where the quantities $\overset{4}{\mathbf{B}}_e$, $\mathbf{G}_e$ and $\partial\phi/\partial\mathbf{S}^S$ can be substituted by (5.2.4) and (5.3.3). The second terms of the material tensors $\overset{4}{\mathbf{B}}$ and $\mathbf{G}$ only appear for ideal-plastic materials in case of loading. For brittle materials (e.g. concrete under tension) and ideal-plastic materials in case of unloading we have $\overset{4}{\mathbf{B}} = \overset{4}{\mathbf{B}}_e$ and $\mathbf{G} = \mathbf{G}_e$.

6. FINAL REMARKS

In this paper it has been shown that the balance equations and the thermodynamical restrictions lead to Darcy's well-known filter law and Fick's diffusion laws as well as to constitutive equations for fluid- and gas-filled elasto-plastic porous solids. The constitutive equation of the solid skeleton as well as the yield condition contain the hydrostatic pressure imposed on the whole body as a part of the stress tensor $\mathbf{S}^S$. An extension of the presented constitutive equations to those physical situations where constitutive interaction of the media solid skeleton, fluid and gas plays a dominant role is possible.

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